

McGill/94-21  
 (May, 1994)  
 hep-th/9409083

# STRING ORGANIZATION OF FIELD THEORIES: DUALITY AND GAUGE INVARIANCE

Y.J. Feng<sup>†</sup> and C.S. Lam<sup>\*</sup>

Department of Physics, McGill University, 3600 University St.  
 Montreal, P.Q., Canada H3A 2T8

## Abstract

String theories should reduce to ordinary four-dimensional field theories at low energies. Yet the formulation of the two are so different that such a connection, if it exists, is not immediately obvious. With the Schwinger proper-time representation, and the spinor helicity technique, it has been shown that field theories can indeed be written in a string-like manner, thus resulting in simplifications in practical calculations, and providing novel insights into gauge and gravitational theories. This paper continues the study of string organization of field theories by focusing on the question of local duality. It is shown that a single expression for the sum of many diagrams can indeed be written for QED, thereby simulating the duality property in strings. The relation between a single diagram and the dual sum is somewhat analogous to the relation between a old-fashioned perturbation diagram and a Feynman diagram. Dual expressions are particularly significant for gauge theories because they are gauge invariant while expressions for single diagrams are not.

## 1 Introduction

Superstring theories enjoy a number of interesting properties at first sight not shared by ordinary four-dimensional field theories. For a superstring theory, (1) the basic entity is a string of size  $O(10^{-32})$  cm, with massless levels and excitation energies  $O(10^{19})$  GeV; (2) loop corrections are ultraviolet finite and quantum gravity is well defined; (3) the fundamental dynamical variables consist of the spacetime  $x^\mu(\sigma, \tau)$  and the internal  $\psi^i(\sigma, \tau)$  fields, all as functions of the worldsheet coordinates  $\sigma$  and  $\tau$ ; (4) these variables propagate as independent free fields throughout the worldsheet, in a manner dependent on the topology but

not on the geometry of the worldsheet (reparametrization and conformal invariance); (5) an external photon of momentum  $p$  and wave function  $\epsilon_\mu(p)$  is inserted into the string through a vertex operator  $\epsilon(p) \cdot [\partial_\tau x(\sigma, \tau)] \exp[ip \cdot x(\sigma, \tau)]$ , a form which is fixed by conformal invariance; (6) conformal invariance leads to local (Veneziano) duality of the scattering amplitude. A scattering process is described by one or very few string diagrams. In particular, for elastic scattering in the tree approximation, one string diagram (the Veneziano amplitude) gives rise simultaneously to all the  $s$ -channel and the  $u$ -channel exchanges.

In contrast, in ordinary four-dimensional field theories (QFT), (1') the basic entities are point particles; (2') loop corrections contain ultraviolet divergences and quantum gravity is non-renormalizable; (3') the fundamental dynamical variables are fields  $\psi^i(x)$  of the four-dimensional spacetime coordinates  $x^\mu$ ; (4') these fields propagate freely only between vertices, where interactions take place; (5') external photons are inserted into a Feynman diagram through the photon operator  $\epsilon(p) \cdot A(x) \exp[ip \cdot x]$ ; (6') there are many distinct Feynman diagrams contributing to a scattering amplitude. For elastic scattering,  $s$ - and  $u$ -channel exchanges are given by different diagrams that must be added up together.

In the 'low-energy' limit when  $E \ll 10^{19}$  GeV, a string of dimension  $10^{-32}$  cm is indistinguishable from a point, energy levels  $O(10^{19})$  GeV are too high to matter, so one would expect a string theory to reduce to an ordinary field theory of zero masses. In fact, explicit calculations have been carried out [1, 2] to show that a one-loop  $n$ -gluon amplitude in a string theory does reduce to corresponding results in field theory. In order to go beyond one loop or the  $n$ -gluon amplitude, where no simple string expression is available, it is better to proceed in a different way. Since the string properties (3)–(6) are quite different from the field-theory properties (3')–(6'), the equivalence of string and QFT at low energies is not immediately obvious. The purpose of this series of papers is, among other things, to make these connections.

There are two motivations for doing so. At a theoretical level, one can hope to gain new insights into gauge and gravitational theories from the string arrangement. For example, according to (3) and (3'), spacetime and internal coordinates are treated on an equal footing in a string, but not so in QFT. This asymmetry makes *local* gauge transformations in QFT awkward to deal with and practical calculations difficult to obtain. It would therefore be nice to be able to reformulate gauge and a gravitational theories in the symmetrical way of a string. Conversely, one can hope that the successful multiloop treatment of QFT, when written in a string-like manner, can give hints useful for multiloop string calculations. On the practical side, string-like organization of an field-theoretical amplitude allows the spinor helicity technique [3, 4] to be used on multiloop diagrams as easily as for tree diagrams [5]; it also enables color, spin, and momentum degrees of freedom to propagate independently, a separation that leads to efficient simplifications in actual calculations, so much so that amplitudes not computable by ordinary means can be obtained when organized in this novel way. Examples of this includes the Parke-Taylor  $n$ -gluon amplitude [6], the factorization of

identical-helicity photons produced in the  $e^+e^-$  annihilation into  $\mu$ -pairs [4], the computation of the one-loop  $n$ -photon amplitude of identical helicities [7], and the calculation of the  $n$ -gluon one-loop amplitudes [1, 2, 8].

In the low energy limit, the variable  $\sigma$  along the string is frozen, so the dynamical fields in (3) are effectively functions of  $\tau$  alone. To convert (3') to (3), one must be able to free the dependence of  $\psi^i$  on  $x$ , to make them both independent functions of some *proper time*  $\tau$ . This can be accomplished by using the Schwinger-parameter representation for the field-theoretic scattering amplitude [5]. In this representation, every vertex  $i$  in a Feynman diagram is assigned a proper time  $\tau_i$ , and each propagator  $r$  is assigned a Schwinger parameter  $\alpha_r$ . If  $r = (ij)$  is a line connecting vertices  $i$  to  $j$ , then  $\alpha_r = |\tau_j - \tau_i|$ . If we regard the Feynman diagram as an electric circuit with resistances  $\alpha_r$ , then  $x(\tau_i)$  can be interpreted as (the four-dimensional) voltage at the vertex  $i$  [9], thus freeing it to be an independent function of  $\tau$ . Spacetime flow is thus analogous to the current flow of an electric network, or equivalently the change of the electrical potential from point to point.

Color flows can be isolated by creating color-oriented vertices in Feynman diagrams [5]. One color-oriented vertex may be related to another by twisting, thus creating twisted Feynman diagrams in much the same way like twisted open strings. The color subamplitudes so isolated with these color-flow factors are gauge invariant.

With massless fermions, chirality and helicities are conserved and this allows free and unobstructed spin flows. This is the essence of the spinor helicity technique which is applicable to photons and gluons as well, for a spin-1 particle can kinematically be regarded as the composite of two spin- $\frac{1}{2}$  particles. The Schwinger proper-time formalism allows the spinor helicity technique to continue to work in loop diagrams [5]; otherwise the loop momentum would be there to break chirality conservation and to obstruct the free flow of spins.

The spacetime, color, and spin flows thus obtained approximate the properties (3) and (4) of the string. Moreover, using *differential circuit identities* [5], external electromagnetic vertices can be converted from (5') to (5). What remains to be considered in completing the string organization of field theory is then the property of *local duality*. Historically it was this unusual feature, first seen in the Veneziano model [10], that marked the beginning of string theory. To what extent this interesting property is retained in field theory is therefore an interesting topic to study. We propose to make a first attempt in that direction in the present paper.

Duality for tree diagrams is considered in Sec. 2, where its exact meaning in field theory is also discussed. For the sake of simplicity we shall confine that section to a scalar theory where a scalar photon field  $A$  is coupled to a charged scalar meson field  $\phi$  via the Lagrangian  $e\phi^*\phi A$ , but it turns out that once we solve the duality problem here it is solved in quantum electrodynamics as well. To prepare for discussions of QED and multiloop amplitudes, we review in Sec. 3 the Schwinger-parameter representation for a field-theoretic scattering amplitude which provides the main tool for further discussions. Duality for the multiloop

scalar theory is discussed in Sec. 4, and duality for QED is discussed in Sec. 5. The problem of QCD is much harder and we shall defer that to a future publication. Finally, a concluding section appears in Sec. 6.

## 2 Duality for Tree Diagrams in a Scalar-Photon Theory

Duality in a string theory follows from its conformal invariance, which in the special case of a four-point amplitude can be used to fix the (complex) worldsheet positions of three of the external lines to be 0, 1 and  $\infty$ , and the fourth one to be  $x \in [0, 1]$ . A particularly simple example is the Veneziano amplitude [10] in the Mandelstam variables  $s$  and  $u$ ,

$$\begin{aligned} A(s, u) &= -B(-u, -s) = -\int_0^1 x^{-u-1} (1-x)^{-s-1} dx \\ &= -\frac{\Gamma(-s)\Gamma(-u)}{\Gamma(-s-u)} . \end{aligned} \quad (2.1)$$

This amplitude has poles when either  $s$  or  $u$  is a non-negative integer (in units of  $[M_P \sim O(10^{19}) \text{ GeV}]^2$ ), but there are no simultaneous  $s$ - and  $u$ -channel poles. The amplitude can be expanded either as a sum of  $s$ -channel poles, represented purely by  $s$ -channel exchange Feynman diagrams, *or* a sum of  $u$ -channel poles, represented purely by  $u$ -channel exchange diagrams. There is no need to *add* both the  $s$ -channel and the  $u$ -channel diagrams, as is necessary in ordinary quantum field theories. The  $u$ -channel amplitude is obviously equal to the  $s$ -channel amplitude, and both are equal to the single integral in (2.1). This is *duality*.

These results can be obtained directly from a pole expansion of the Euler Gamma functions, or from the integral representation for the Beta function. In the latter case, an expansion of the integrand about  $x = 0$  gives rise to the  $u$ -channel poles, and an expansion of the integrand about  $x = 1$  gives rise to the  $s$ -channel poles.

At present energies,  $|s|, |u| \ll 1$  (in units of  $M_P^2$ ), so only the massless poles contribute, giving rise to

$$A(s, u) \simeq \frac{1}{s} + \frac{1}{u} . \quad (2.2)$$

The  $u$ -channel pole comes from the divergence of the integral near  $x = 0$  when  $u = 0$ , and the  $s$ -channel pole comes from the divergence of the integral near  $x = 1$  when  $s = 0$ . In this form, the amplitude does not *appear* to be ‘dual’ anymore, because *both* the  $s$ -channel and the  $u$ -channel poles are *summed*, instead of having a *single* expression like (2.1), where only a sum of the  $s$ -channel *or* a sum of the  $u$ -channel poles are present. Nevertheless, appearances are deceiving, because (2.2) follows mathematically from (2.1), which is dual.

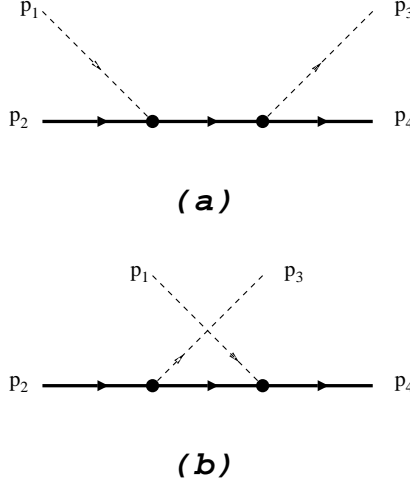


Figure 1: Compton scattering diagrams, in which  $s = (p_1 + p_2)^2$  and  $u = (p_2 - p_3)^2$

In other words, the  $u$ -channel poles in (2.2) can be formally obtained by an infinite sum of massive  $s$ -channel poles, and (2.2) is as dual as it can be at low energies.

It is instructive for later discussions to obtain (2.2) directly from the integral representation of (2.1). For that purpose, divide the integral in  $x$  into two halves at the midpoint  $x = \frac{1}{2}$ . Since  $|s|, |u| \ll 1$ , the contribution of the integral comes mainly from  $x = 0$ , so we can put  $(1 - x)^{-s-1} \simeq 1$  there. Similarly the term  $x^{-u-1}$  can be ignored in the second integral. Next, make the transformation  $y = \ln(2x)$  in the first integral, so that  $y \in [-\infty, 0]$  there, and the transformation  $y = -\ln[2(1 - x)]$  in the second integral, so that  $y \in [0, \infty]$  there. Then for  $|s|, |u| \ll 1$ , the integral in (2.1) becomes

$$\begin{aligned}
 A(s, u) &\simeq - \int_{-\infty}^0 dy \exp(-uy) - \int_0^{\infty} dy \exp(sy) \\
 &= \frac{1}{u} + \frac{1}{s},
 \end{aligned} \tag{2.3}$$

which is the same as (2.2).

If one were to start directly from a massless scalar field theory  $\phi^* \phi A$  (where all fields are scalar and massless), then the ‘Compton scattering’ amplitude given by Fig. 1 is identical to (2.2). In that sense the field-theoretic amplitude is already dual, or as dual as it can be at the present energy range. One might still be unsatisfied with this remark about duality, and points out that the original dual amplitude in (2.1) is given by a *single* integral, whereas in (2.3) this is given by the sum of *two* integrals. Since (2.3) comes from (2.1) it must be able to write it as a single integral as well. All that we have to do is to define a function

$P = \theta(-y)(-uy) + \theta(y)(sy)$ , then

$$A(s, u) = - \int_{-\infty}^{\infty} dy \exp(P) , \quad (2.4)$$

which is of course equivalent to the Veneziano integral (2.1) at the present energy range. We shall refer to expressions of this type, where the sum of a *number of diagrams* is represented by a *single* (possibly multi-dimensional) integral, as *dual expressions*. What allows the dual expression (2.4) to be written is not so much the explicit form of the integrands in (2.3), but that the two integrals there have non-overlapping ranges in  $y$ . Given that, it is always possible to define a common integrand  $\exp(P)$  so that the two integrals can be combined into one.

Similar reasoning shows that dual expressions can be written for other processes in a scalar-photon theory. Since we are imitating the low energy limit of strings, we may simplify writings by assuming all particles to be massless. In that case, the Schwinger-parameter representation for a scalar propagator is

$$\frac{1}{q^2 + i\epsilon} = -i \int_0^{\infty} d\alpha \exp(i\alpha q^2) , \quad (2.5)$$

and the variable  $\alpha$  is called a Schwinger proper-time parameter. Using this, we can obtain (2.3) from (2.5) simply by letting  $i\alpha = -y$ ,  $q^2 = u$  in the first term, and  $i\alpha = y$ ,  $q^2 = s$  in the second term. The fact that we can get the Veneziano integral representation (2.3) from the Schwinger representation confirms the claim that the Schwinger-parameter representations are string-like.

Consider now a more complicated example, Fig. 2, in which charged particles scatter to produce  $m$  (scalar) photons from one charged line and  $n$  photons from another. We have drawn only one of the  $(m+1)!(n+1)!$  possible diagrams; others are obtained from it by permuting the photon lines.

Let us first establish some common notations. Assign to each vertex of a Feynman diagram a *proper time*  $\tau$ , as illustrated in Fig. 2. The Schwinger proper-time parameters  $\alpha$  are then given by differences of the proper times. Specifically, if  $r = (ij)$  is an internal line between vertices  $i$  and  $j$ , then  $\alpha_r = |\tau_i - \tau_j|$ . In this way, all the proper-time differences are determined but translational invariance prevents the origin of proper time to be fixed. Let

$$\begin{aligned} \int_a^b d\tau_{[12\dots n]} &\equiv \int_a^b d\tau_1 \int_{\tau_1}^b d\tau_2 \cdots \int_{\tau_{n-1}}^b d\tau_n , \\ \int_a^b d\tau_{[12(345)678]} &\equiv \int_a^b d\tau_{[12]} \int_{\tau_2}^b d\tau_{[678]} \left( \prod_{i=3}^5 \int_{\tau_2}^{\tau_6} d\tau_i \right) , \\ \langle \int d\tau_{[12\dots n]} \rangle &\equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\tau_{[12\dots n]} , \end{aligned}$$

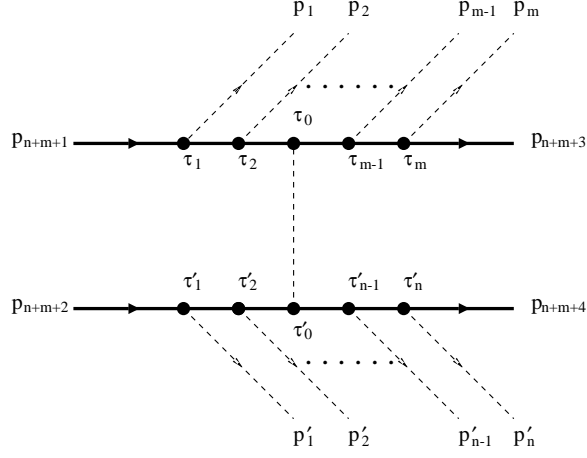


Figure 2: A tree diagram for multiphoton emission from charged-particle scattering

$$\begin{aligned}
\langle \int d\tau_{[12(345)678]} \rangle &\equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T d\tau_{[12(345)678]} , \\
\langle \int d\tau_{(12\dots n)} \rangle &\equiv \langle \int d\tau_{[(12\dots n)]} \rangle .
\end{aligned} \tag{2.6}$$

In short, the integration variable enclosed between square brackets are ordered, and those between round brackets are unordered.

We can now return to Fig. 2. Its amplitude is

$$A = (-i)^{n+m+2} \langle \int d\tau_{[12\dots 0\dots m]} \rangle \langle \int d\tau'_{[12\dots 0\dots n]} \rangle \exp(iP) , \tag{2.7}$$

for some quadratic function  $P$  of the external momenta obtained by using (2.5) on each propagator. The detailed form of  $P$  does not concern us at the moment. The other diagrams are obtained by permuting the photon lines in Fig. 2, so their amplitudes are all given by something like (2.7), but with the  $\tau$  and  $\tau'$  integration regions permuted separately. The detailed form of the quadratic function  $P$  may change from region to region but again we do not have to worry about it. Since the integration regions of these different diagrams do not overlap, it is possible to define a common  $P(\tau, \tau')$  equal to the individual  $P$ 's in their respective regions. In this way, all the  $(m+1)!(n+1)!$  diagrams can be summed up to get a single dual expression

$$A_{sum} = (-i)^{n+m+2} \langle \int d\tau_{(012\dots m)} \rangle \langle \int d\tau'_{(012\dots n)} \rangle \exp(iP) , \tag{2.8}$$

in which the integration regions are completely unordered.

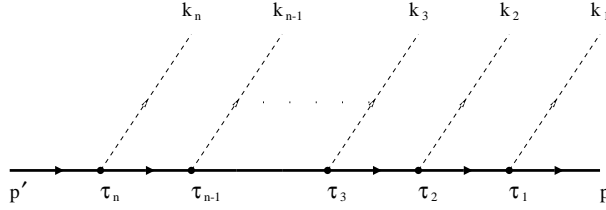


Figure 3: Multiphoton emission from a charged line

The reasoning can obviously be extended to any tree diagram in a scalar-photon theory. In every case, each charged line provides a platform for ordering the photon lines attached to it. Different diagrams correspond to different permutations of these photon lines, so they correspond to different integration regions in the proper times. A sum into a single dual expression in which the proper time integration regions are unordered is clearly possible. Furthermore, with minor modifications to be discussed in Sec. 4, essentially the same consideration works for multiloop diagrams as well.

There are three remarks to be made. First of all, it is interesting to note that the relation between a dual expression and a Feynman diagram is very much like the relation between a Feynman diagram and an old-fashioned diagram. Recall that a Feynman diagram with  $n$  vertices is made up of a sum of  $n!$  old-fashioned diagrams, corresponding to the  $n!$  possible (real) time orderings of its vertices. Similarly, a dual ‘diagram’ consists of a sum of Feynman diagrams, and they differ from one another by the *proper-time* orderings of their vertices.

Secondly, proper-time orderings in QED diagrams are very simple and natural, because there are the conserved charged lines along which the photon vertices can be proper-time ordered. In contrast, in a neutral scalar  $\phi^3$  theory, or in a pure gluon QCD, all lines are equivalent and there are no obvious ways to proper-time order the vertices, especially in multiloop diagrams. This is one of the difficulties one encounters in QCD.

Thirdly, there is the question of how useful these dual expressions are. That naturally depends on the details of the diagrams one tries to sum, and how simple the resulting integrand  $\exp(iP)$  is. For example, consider the emission of scalar photons shown in Fig. 3, where  $p$  and  $k_i$  are massless but  $p'$  may be offshell. Using (2.5), and expressing the Schwinger parameters  $\alpha$  as differences of the proper-time parameters  $\tau_i$ , the exponent  $P$  in the integrand  $\exp(iP)$  becomes

$$P = \sum_{i=1}^{n-1} \alpha_i (p + \sum_{j=1}^i k_j)^2$$



$$= 2 \sum_{i=1}^n (\tau_i - \tau_n) p \cdot k_i + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\tau_j - \tau_n) k_i \cdot k_j . \quad (2.9)$$

By using momentum conservation, this can also be written in a more symmetric form:

$$P = (\tau_1 - \tau_n) p' \cdot p + \sum_{i=1}^n (\tau_i - \tau_n) p' \cdot k_i - \sum_{i=1}^n (\tau_1 - \tau_i) p \cdot k_i - \frac{1}{2} \sum_{i \neq j=1}^n |\tau_i - \tau_j| k_i \cdot k_j . \quad (2.10)$$

The function  $P$  for a permuted diagram can be obtained from these expressions by permuting the photon momenta.

In general, it is quite impossible to obtain a closed analytic expression for its dual sum  $A_{sum}$ . However, in the eikonal approximation where the photon momenta are considered small,  $O(k_i \cdot k_j)$  terms can be neglected from (2.9), then

$$P \simeq 2 \sum_{i=1}^n (\tau_i - \tau_n) p \cdot k_i . \quad (2.11)$$

The amplitude for Fig. 3 is then proportional to

$$A = (-i)^{n-1} \int_{\tau_n}^{\infty} d\tau_{[n-1, n-2, \dots, 2, 1]} \exp(iP) , \quad (2.12)$$

where  $\tau_n$  is completely arbitrary. This freedom can be exploited to render the amplitude  $A$  more symmetrical, if we multiply and divide it by

$$2p \cdot \left( \sum_{i=1}^n k_i \right) = -i \int_0^{\infty} d\tau_n \exp \left( i\tau_n 2p \cdot \sum_{j=1}^n k_j \right) . \quad (2.13)$$

Then

$$A = 2p \cdot \left( \sum_{i=1}^n k_i \right) (-i)^n \int_0^{\infty} d\tau_{[n, n-1, \dots, 1]} \exp(i\tilde{P}) , \quad (2.14)$$

with

$$\tilde{P} = 2 \sum_{i=1}^n \tau_i p \cdot k_i . \quad (2.15)$$

Since  $\tilde{P}$  is completely symmetrical in all the  $k_i$ , it is identical in all the permuted diagrams, so the dual sum of the  $n!$  permuted diagram is

$$\begin{aligned} A_{sum} &= 2p \cdot \left( \sum_{i=1}^n k_i \right) (-i)^n \int_0^{\infty} d\tau_{(n, n-1, \dots, 1)} \exp(i\tilde{P}) \\ &= 2p \cdot \left( \sum_{i=1}^n k_i \right) \prod_{j=1}^n \frac{1}{2p \cdot k_j} , \end{aligned} \quad (2.16)$$

which is the well known eikonal expression.

Looking at this example, we see that there are two important ingredients to make it successful. The first is that  $\tilde{P}$  has an identical functional form in every one of the  $n!$  integration regions. We shall refer to integrand of this kind, that it has the same functional form in all integration regions, to be *symmetrical*. The second ingredient is that the final integrals in (2.16) are simple enough to be computed analytically. It turns out that the first ingredient is relatively easy to come by. This is because the functional form of  $P$  can be altered, either by using momentum conservation to substitute one external momentum by the negative sum of all others, or by changing something like  $\tau_i - \tau_j$  to  $(\tau_i - \tau_k) + (\tau_k - \tau_j)$ . Quite often by making these changes one can manipulate  $P$  into a symmetric form. For example, eq. (2.10) is completely symmetrical in the indices  $i = 2$  to  $n - 1$  although (2.9) is not. Yet, without the second ingredient, there is really not much point in achieving the first. This is clearly seen by comparing (2.10) with (2.9). Without the eikonal approximation, it is impossible to obtain  $A_{sum}$  in either form, in spite of the symmetry of (2.10). Even numerically it is not clear that a symmetric  $P$  is easier to compute, especially if its functional form is forced to be very complicated when it is made symmetric.

### 3 Schwinger-parameter Representation

Every scattering amplitude can be written in the form

$$A = \left[ \frac{-i\mu^\epsilon}{(2\pi)^d} \right]^\ell \int \prod_{a=1}^\ell (d^d k_a) \frac{S_0(q, p)}{\prod_{r=1}^N (-q_r^2 + m_r^2 - i\epsilon)} , \quad (3.17)$$

where  $d = 4 - \epsilon$  is the dimension of spacetime,  $k_a$  ( $1 \leq a \leq l$ ) are the loop momenta,  $q_r, m_r$  ( $1 \leq r \leq N$ ) are the momenta and masses of the internal lines, and  $p_i$  ( $1 \leq i \leq n$ ) are the outgoing external momenta. Since we will be mainly interested in massless field theories, all  $m_r^2$  will be set equal to 0 in the following.

The numerator function  $S_0(q, p)$  contains everything except the denominators of the propagators. Specifically, it is the product of the vertex factors, numerators of propagators, wave functions of the external lines, symmetry factor, and the signs associated with closed fermion loops. All the  $i$ 's and  $(2\pi)$ 's have been included in the factor before the integral.

By introducing a Schwinger proper-time parameter  $\alpha_r$  for each internal line to represent its scalar propagator as in (2.5), the loop integrations in (3.1) can be explicitly carried out to obtain the Schwinger-parameter representation [9]

$$A = \int [D\alpha] \Delta(\alpha)^{-d/2} S(q, p) \exp[iP] , \quad (3.18)$$

where

$$\int [D\alpha] \equiv \left[ \frac{(-i)^{d/2} \mu^\epsilon}{(4\pi)^{d/2}} \right]^l i^N \int_0^\infty \left( \prod_{r=1}^N d\alpha_r \right) ,$$

$$P = \sum_{r=1}^N \alpha_r q_r^2 \equiv \sum_{i,j=1}^n Z_{ij}(\alpha) p_i \cdot p_j , \quad (3.19)$$

$$S(q, p) \equiv \sum_{k \geq 0} S_k(q, p) . \quad (3.20)$$

In spite of the same notation, the momenta  $q_r$  in (3.2) to (3.4) cannot be the same as the one in (3.1) since loop-momentum integrations have now been carried out. Instead, it is to be interpreted as the current flowing through the  $r$ th line of an electric circuit given by the Feynman diagram, where  $p_i$  are the outgoing currents and  $\alpha_r$  are the resistances of the  $r$ th line. With this interpretation,  $P$  in (3.2) and (3.3) becomes the power consumed by the circuit, and  $Z_{ij}$  is then the impedance matrix. On account of current conservation,  $\sum_{i=1}^n p_i = 0$ ,  $q_r$ ,  $P$ , and hence the amplitude  $A$  are invariant under the *level transformation*  $Z_{ij} \rightarrow Z_{ij} + \xi_i + \xi_j$  for any arbitrary  $\xi_i$ . This enables us to choose an impedance matrix with  $Z_{ii} = 0$ . Some of the formulas below, including (3.7), (3.11), and (3.16), are not valid without this condition. Unless otherwise specified, this is a choice we will adopt throughout. Note that these and the following formulas are equally valid for tree diagrams, or a combination of trees and loops.

Suppose the numerator function  $S_0(q, p)$  is a polynomial in  $q$  of degree  $j$ . Then  $S_k(q, p)$  is defined to be a polynomial in  $q$  of degree  $j - 2k$ , obtained from  $S_0(q, p)$  by contracting  $k$  pairs of  $q$ 's in all possible ways and summing over all the contracted results. The rule for contracting a pair of  $q$ 's is:

$$q_r^\mu q_s^\nu \rightarrow -\frac{i}{2} H_{rs}(\alpha) g^{\mu\nu} \equiv q_r^\mu \sqcup q_s^\nu . \quad (3.21)$$

The circuit quantities in (3.2) to (3.5), including  $\Delta$ ,  $q_r$ ,  $P$ ,  $Z_{ij}$  and  $H_{rs}$ , can all be obtained directly from the Feynman diagram [9]. For example, the formula for the impedance matrix and the function  $\Delta$  are

$$\Delta = \sum_{T_1} \prod \alpha , \quad (3.22)$$

$$Z_{ij} = -\frac{1}{2} \Delta^{-1} \sum_{T_2^{ij}} \prod \alpha . \quad (3.23)$$

These formulas have the following meaning. An  $l$ -loop diagram can be turned into a tree by cutting  $l$  appropriate lines.  $\Delta$  in (3.6) is obtained by summing over the set  $T_1$  of all such

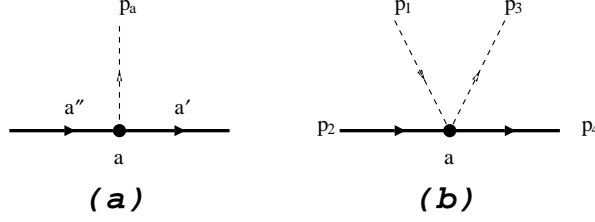


Figure 4: Cubic vertex  $C_a$  and seagull vertex  $Q_a$  of scalar electrodynamics

cuts, with the summand being the product of the  $\alpha$ 's of the cut lines in each case. For tree diagrams where  $l = 0$ , by definition  $\Delta = 1$ . Similarly, let  $T_2^{(ij)}$  be the set of all cuts of  $l + 1$  lines so that the diagram turns into two disconnected trees, with vertex  $i$  in one tree and vertex  $j$  in another. Then  $-2\Delta \cdot Z_{ij}$  in (3.7) is given by the sum over  $T_2^{(ij)}$ , with the summand being the product of  $\alpha$ 's of the cut lines.

Besides satisfying Kirchhoff's law, the electric-circuit quantities obey a number of *differential circuit identities* [5]:

$$\frac{\partial}{\partial \alpha_r} P(\alpha, p) = q_r^2, \quad (3.24)$$

$$\frac{\partial}{\partial \alpha_s} q_r(\alpha, p) = H_{rs} q_s, \quad (3.25)$$

$$\frac{\partial}{\partial \alpha_t} H_{rs}(\alpha) = H_{rt}(\alpha) H_{ts}(\alpha). \quad (3.26)$$

Moreover, the *contraction function*  $H_{rs} = H_{sr}$  is 'conserved' at each vertex as if the external currents were absent, *i.e.*, if  $\sum_{r \in V} q_r = p$  is obeyed at some vertex, then  $\sum_{r \in V} H_{rs} = 0$  for all  $s$ . In particular, if  $q_r$  does not involve  $\alpha$ 's as is the case when it is a branch of a tree, then  $H_{rs} = 0$  for all  $s$ .

So far all quantities are expressed as functions of  $p_i$  and  $\alpha_r$ . As in the case for tree diagrams, we can assign each vertex with a proper time  $\tau_i$  and consider  $\alpha_r = |\tau_i - \tau_j|$  if  $r = (ij)$ . We can then convert all  $\alpha$ -integrations into  $\tau$ -integrations.

In scalar QED, there are two kinds of vertices: the cubic vertex  $C_a = e\epsilon(p_a) \cdot (q_{a'} + q_{a''})$  in Fig. 4(a) and the seagull vertex  $Q_a = 2e^2\epsilon(p_1) \cdot \epsilon(p_3)$  in Fig. 4(b). We shall call a cubic vertex 'external', and perhaps less confusingly of *type-a*, if it consists of one external photon line and two internal charged-scalar lines. A type-a vertex  $a$  has a string-like representation [5]

$$C_a = -ie\epsilon(p_a) \cdot D_a(iP), \quad (3.27)$$

where

$$D_a P \equiv \partial_a \frac{\partial P}{\partial p_a} , \quad \partial_a \equiv \frac{\partial}{\partial \tau_a} . \quad (3.28)$$

Unfortunately, the same representation is not true for internal cubic vertices.

To make (3.12) useful, we shall define  $D_a$  to operate on functions of the form  $f(\tau) \prod_r q_r \exp(iP)$  like a derivation, i.e., like a first derivative satisfying the product rules but not like a second derivative. To complete the definition, we must also define  $D_a$  when it operates on  $f(\tau)$ ,  $P$  and  $q_r$ . For each of these three elementary operations, it will be  $D_a = \partial_a (\partial / \partial p_a)$  as in (3.12). This leads to  $D_a f(\tau) = 0$  and it can be shown that [5]

$$D_a q_r = H_{ar} , \quad D_a^\mu D_b^\nu P = 2g^{\mu\nu} H_{a'b'} , \quad (3.29)$$

where  $b$  is another type-a vertex, and  $a' \neq r \neq a''$ .

Eq. (3.11) makes it possible to replace a vertex  $C_a$  by an operation involving  $D_a$ . Eq. (3.13) shows that the necessary contractions (3.5) can automatically be accommodated as well. Consequently, a scalar QED amplitude with  $n_a$  vertices of type-a can be written as [5]

$$A = \int [D\alpha] \Delta^{-d/2} (-ie)^{n_a} [\epsilon(p_1) \cdot D_1] [\epsilon(p_2) \cdot D_2] \cdots [\epsilon(p_{n_a}) \cdot D_{n_a}] S^{int}(q, p) \exp(iP), \quad (3.30)$$

where  $S^{int}(q, p)$  is given by the product of the non-type-a vertices and terms generated from their mutual contractions. Moreover, it is true that [5]

$$\partial_a \Delta = 0 \quad (3.31)$$

for a type-a vertex, so it does not matter whether  $\Delta$  in (3.14) is put before or after the  $D_a$ 's. Another useful relation to know is

$$\partial_a Z_{ij} = 0 \quad (3.32)$$

provided  $i \neq a \neq j$ . This relation can be used to show how the string-like vertex changes under a gauge transformation, when  $\epsilon(p_a)$  is replaced by  $p_a$ . In that case, remembering that  $Z_{aa} = 0$ , (3.3) and (3.16) give

$$\begin{aligned} C_a &\rightarrow -ie \partial_a \left( p_a \cdot \frac{\partial P}{\partial p_a} \right) = -2ie \partial_a (p_a \cdot \sum_i Z_{ai} p_i) \\ &= -ie \partial_a \sum_{i,j} Z_{ij} p_i \cdot p_j = -ie \partial_a P . \end{aligned} \quad (3.33)$$

## 4 Multiloop Duality for the Scalar Theory

In the scalar-photon theory with interaction  $\phi^* \phi A$ , the most general amplitude is given by (3.2), with  $S = 1$  and  $P$  given by (3.3) and (3.7). The Schwinger parameters  $\alpha$  will be

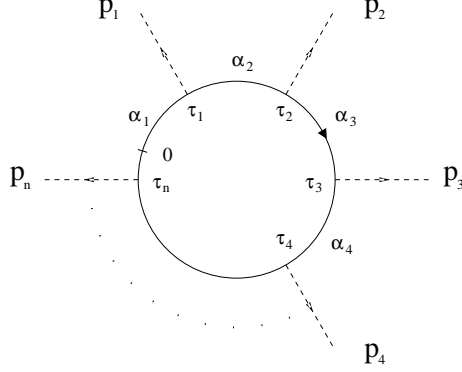


Figure 5: An  $n$ -photon one-loop diagram

expressed as differences of the proper-time parameters  $\tau$ , and the proper times will again be ordered along the charged lines as in the tree cases. The only new problem here is where to begin the ordering in the case of a charged-scalar loop.

Since the origin of the proper time is never determined by the  $\alpha$ 's, it can be chosen arbitrarily, say at the position marked '0' in Figs. 5 and 6. We must now insert into (3.2) a factor

$$1 = \int_0^\infty dT \delta(\sum_{loop} \alpha - T) \quad (4.34)$$

for every charged loop, where the sum is taken over all the  $\alpha$ 's in the loop. So for Fig. 5, the  $\alpha$ -integrations can be replaced by

$$\prod_{i=1}^n \left( \int_0^\infty d\alpha_i \right) = \int_0^\infty \frac{dT}{T} \int_0^T d\tau_{[12\dots n]} \equiv \langle \int d\tau_{[12\dots n]} \rangle, \quad (4.35)$$

where  $\alpha_i = \tau_{i+1} - \tau_i$  for  $i \leq n-1$ , and  $\tau_n = T - (\tau_n - \tau_1)$ . Strictly speaking, there is an inconsistency in the definition of  $\langle d\tau_{[...]} \rangle$  between (4.2) and (2.6), but in practice (4.2) is always used for closed charged loops and (2.6) is always used for open charged lines.

One can now obtain dual amplitudes by summing over all photon permutations in exactly the same way as before. For example, for Fig. 5 and its permuted diagrams, the sum is proportional to

$$A_{sum} = \langle \int d\tau_{(12\dots n)} \rangle \Delta^{-d/2} \exp(iP). \quad (4.36)$$

For Fig. 6 and its permuted diagrams, the sum is proportional to

$$A_{sum} = \langle \int d\tau_{(12\dots m+a)} \rangle \langle \int d\tau'_{(12\dots n+b)} \rangle \langle \int d\tau''_{(12\dots a+b+c)} \rangle \Delta^{-d/2} \exp(iP). \quad (4.37)$$

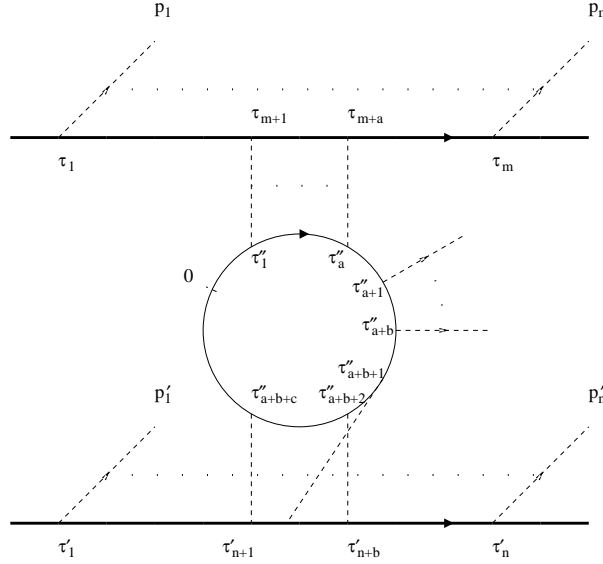


Figure 6: A complicated multiloop scattering diagram

As in tree amplitudes, how useful such dual expressions are depends on the complexity of  $P$  and  $\Delta$  in each case. In the eikonal approximation (2.11)—(2.16), the integrals can be carried out because  $\tilde{P}$  is common for all integration regions *and* because it has a simple dependence on the proper times. Now the first condition is not hard to meet, if the diagrams to be summed have a high degree of symmetry. For example, for the well-studied case of Fig. 5 [8, 11, 5], eqs. (3.6) and (3.7) show that  $\Delta = \sum_{i=1}^n \alpha_i = T$ , and

$$Z_{ij} = -|\tau_i - \tau_j| (T - |\tau_i - \tau_j|) / 2T \equiv G_B(\tau_i, \tau_j) , \quad (4.38)$$

which has a symmetric form in the sense that it has the same functional form in all integration regions. However, one is still unable to evaluate the integral (4.3) analytically because of its relatively complicated  $\tau$ -dependences.

For the reason discussed at the end of Sec. 2,  $P$  can be made symmetric or partially symmetric in many cases. One way of seeing this is the following. If we modify Fig. 3 by adding on a charged-scalar propagator at each end, then analogous to (2.10) one can produce a form of  $P$  which is completely symmetrical in all the photon lines. We have already seen that the  $P$  in Fig. 5 is completely symmetrical in all its photon lines as well. Now every Feynman diagram can be built up from a number of open charged lines with its attached photons, and a number of one-charged-loop diagrams with its attached photons, by joining together pairs of photon lines. Mathematically, one obtains the resulting amplitudes by multiplying these  $P$ 's of the components, together with the propagators of the joined photon lines in the form of (2.5), then carries out the momentum integrations of the joined photon lines. Since the dependences on these joined momenta are Gaussian, such integrations can

be carried out, and one again obtains a result of the form (3.2), with  $S(q, p) = 1$ , and with  $P$  of (3.2) a function of the component  $P$ 's. Since the component  $P$ 's are symmetric in all the photon lines, they will still be symmetric in the remaining, unjoined, external photon lines, so in this way one can obtain a symmetric form for the final  $P$ . This mechanism for obtaining a symmetric form has been discussed recently [12] in a slightly different language. However, the symmetric form obtained this way is usually much more complicated than those obtained directly from (3.3), (3.6), and (3.7). It is so complicated that it is unlikely to be integrated analytically, nor will it lead to simpler numerical evaluations in most cases. See the end of Sec. 2 for more discussions along these lines. Though one can generally produce other simpler symmetric forms, they are still not simple enough for the integrations to be carried out explicitly. For these reasons there seems to be no particular advantage of having a symmetric form and we will not do so most of the time.

## 5 QED

The most important ingredient for obtaining a dual expression is the presence of a conserved charged line along which to order the interaction vertices. This is independent of the spin of the particles involved, hence one can obtain dual amplitudes in QED just as easily as those in the scalar-photon theory.

The factor  $S(q, p)$  in (3.2) is no longer 1 for QED. For scalar QED, for example, it is made up of the product of the vertex factors  $C_i$  and  $Q_i$  and their contractions. See eq. (3.11) and the paragraph above it. This however makes no difference to the construction of the dual amplitude. Another minor complication is the presence of the seagull vertex  $Q_i$ . This simply adds a Dirac- $\delta$  function contribution to the integrand. For example, in the Compton scattering diagram Fig. 7, all that we have to do to accommodate the seagull vertex in Fig. 7(c) is to define the integrand of the dual amplitude to be

$$\begin{aligned} S \exp(iP) &= \theta(\tau_3 - \tau_1) S_a \exp(iP_a) + \theta(\tau_1 - \tau_3) S_b \exp(iP_b) \\ &+ \delta(\tau_3 - \tau_1) S_c \exp(iP_c) , \end{aligned} \tag{5.39}$$

where  $S_i, P_i$  ( $i = a, b, c$ ) are the respectively factors for the three diagrams Figs. 7(a), 7(b), and 7(c).

What distinguishes the dual expressions of QED from the scalar theory is gauge invariance, in that the diagrams in QED to be summed are connected by gauge invariance. This means that the gauge-dependent parts of each diagram should no longer be there in the dual sum. How the dual expression can be mathematically manipulated to achieve this purpose unfortunately depends on the details. At this moment two general techniques are available to aid us. One is the spinor helicity technique [3, 4, 5], where the reference momenta of



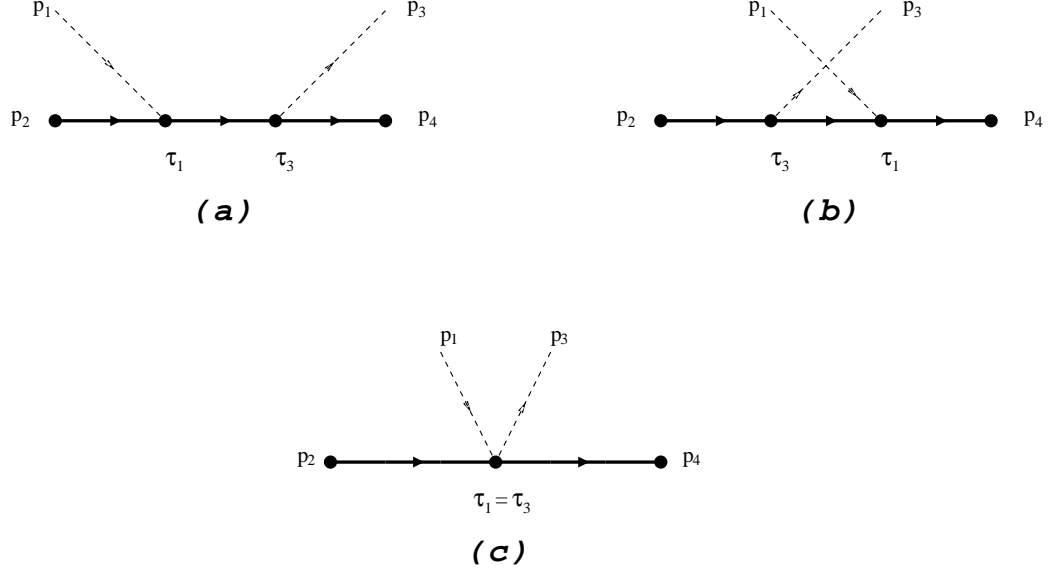


Figure 7: Lowest order Compton scattering diagrams in scalar QED

the photons can be chosen appropriately to reduce the amount of gauge-dependent contributions. The other is the integration-by-parts technique [1, 2, 11] used in connection with the string-like operators appearing in (3.11) and (3.14).

We mentioned previously that the relation between a dual amplitude and a Feynman amplitude is analogous to the relation between a Feynman amplitude and an old-fashioned amplitude. In each case the former is not time-ordered, and the latter is; the only difference being that it is proper-time ordering for the first pair and real-time ordering for the second. Now with gauge theories, there is another parallel between these two cases: an old-fashioned diagram is not relativistically invariant but a Feynman diagram is. Similarly, a Feynman diagram is not gauge invariant but a dual diagram is.

Let us consider two simple examples to illustrate these points. First, consider the Compton amplitude Fig. 7 in scalar QED. A propagator is added to each external charged line so that the vertices 1 and 3 in Figs. 7(a) and 7(b) are of type-a, to enable the string-like vertex (3.11) to be used. Without the seagull term Fig. 7(c), the amplitude is not gauge invariant, so it definitely contains a non-trivial gauge-dependent part. As we shall see below, one can actually manipulate the dual expression so that the seagull vertex seems to disappear, and the gauge-dependent parts from these three diagrams are no longer present.

The expression  $P$  for diagrams (a), (b), and (c) are respectively

$$\begin{aligned}
P_a &= (\tau_3 - \tau_1)(p_1 + p_2) \cdot (p_3 + p_4) + (\tau_1 - \tau_2)p_2 \cdot (p_3 + p_4 - p_1) \\
&\quad + (\tau_4 - \tau_3)p_4 \cdot (p_1 + p_2 - p_3) , \\
P_b &= (\tau_1 - \tau_3)(p_2 - p_3) \cdot (p_4 - p_1) + (\tau_3 - \tau_2)p_2 \cdot (p_3 + p_4 - p_1) \\
&\quad + (\tau_4 - \tau_1)p_4 \cdot (p_1 + p_2 - p_3) , \\
P_c &= (P_a)_{\tau_3=\tau_1} = (P_b)_{\tau_3=\tau_1} .
\end{aligned} \tag{5.40}$$

Using (3.14), the vertex factors in both cases become

$$S_a = S_b = (-ie)^2 [\epsilon(p_1) \cdot D_1] [\epsilon(p_3) \cdot D_3] \equiv S . \tag{5.41}$$

In their respective regions, one can replace  $P_a$  and  $P_b$  by

$$\begin{aligned}
P'_a &= \theta(\tau_3 - \tau_1)P_a \\
P'_b &= \theta(\tau_1 - \tau_3)P_b .
\end{aligned} \tag{5.42}$$

However, since  $\tau$ -differentiations are involved in  $S$  of (5.3),  $S_a \exp(iP'_a)$  and  $S_b \exp(iP'_b)$  are not identical to  $S_a \exp(iP_a)$  and  $S_b \exp(iP_b)$ . It can be checked by explicit calculation that the former already contains the seagull vertex in Fig. 7(c). Hence the dual expression for Fig. 7 is

$$A_{sum} = -ie^2 \langle \int_{\tau_2}^{\tau_4} d\tau_{(13)} \rangle [\epsilon(p_1) \cdot D_1] [\epsilon(p_3) \cdot D_3] \exp(iP') , \tag{5.43}$$

where

$$P' = \theta(\tau_3 - \tau_1)P_a + \theta(\tau_1 - \tau_3)P_b = P'_a + P'_b . \tag{5.44}$$

The fact that the seagull vertex *seems* to have disappeared suggests that we have eliminated the gauge-dependent contributions altogether. To see that explicitly, use (3.17). Then under a gauge transformation,  $\epsilon(p_a) \cdot D_a$  ( $a = 1, 3$ ) is changed into something proportional to  $\partial_a$ . The integral over  $\tau_a$  in (5.5) can then be carried out to yield the boundary contributions at  $\tau_2$  and  $\tau_4$ , and hence the Ward-Takahashi identity. Since there is no trace of explicit cancellations needed at  $\tau_1 = \tau_3$ , it must mean that the gauge-dependent terms in the individual diagrams have now been eliminated.

Another simple example one can mention is QED in the eikonal approximation, Fig. 3. In the soft photon limit for scalar QED, diagrams involving seagull vertex are not dominant because of the presence of one less propagator  $O(k^{-1})$  in the amplitude. The cubic vertex factor is trivial in the soft photon limit, yielding  $S = \prod_{i=1}^n (2e\epsilon(k_i) \cdot p)$ . The dual amplitude can therefore be read off from (2.17) to be

$$A_{sum} = 2p \cdot \left( \sum_{i=1}^n k_i \right) \prod_{j=1}^n \left( \frac{e\epsilon(k_j) \cdot p}{p \cdot k_j} \right) \tag{5.45}$$

It is gauge invariant to leading order in the photon momenta.

## 6 Conclusion

In the Schwinger-parameter representation, QED diagrams differing from one another by the permutation of photon lines correspond to different proper-time ordering of the vertices, and can be formally summed into a single integral over a hypercubic region. This sum is referred to as a dual sum because it is the field-theoretic counter part of a dual amplitude in string theory. The relation between individual Feynman diagrams and their dual sum is analogous to the relation between individual old-fashioned diagrams and their sum into a single Feynman diagram. Among other things, individual Feynman diagrams in QED are not gauge invariant but the dual sum is. Similarly, the individual old-fashioned diagrams are not Lorentz invariant but their sum is. The dual sum allows formal manipulations between different Feynman diagrams to be carried out, *e.g.*, by the integration-by-parts technique on string-like vertices. With appropriate approximations, such as the eikonal approximation, explicit results may sometimes be obtained from the dual expression as well.

Dual expressions for QCD are much more complicated to deal with and are not discussed in the present paper.

## 7 acknowledgement

This research is supported in part by the Natural Sciences and Engineering Research Council of Canada and the Québec Department of Education.s

## References

- [†] e-mail Address: feng@physics.mcgill.ca.
- [\*] e-mail Address: lam@physics.mcgill.ca.
- [1] Z. Bern and D.K. Kosower, Phys. Rev. Lett. **B66** (1991), 1669; Nucl. Phys. **B379** (1992), 451.
- [2] Z. Bern and D.C. Dunbar, Nucl. Phys. **B379** (1992), 562.
- [3] P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, Phys. Lett. **105B** (1981), 215; Nucl. Phys. **B291** (1987), 392; M.L. Mangano and S.J. Parke, Phys. Rep. 200 (1991), 301; R. Gastmans and T.T. Wu, ‘The Ubiquitous Photon’, International Series of Monographs on Physics, Vol. 80 (Clarendon Press, Oxford, 1990).
- [4] Z. Xu, D.-H. Zhang, and L. Chang, Tsinghua University Preprints, Beijing, China, TUTP-84/4, TUTP-84/5, TUTP-84/6.

- [5] C.S. Lam, Nucl. Phys. **B397** (1993), 143; Phys. Rev. **D48** (1993), 873; Can. J. Phys. to appear.
- [6] S. Parke and T. Taylor, Phys. Rev. Lett. 56 (1986), 2459.
- [7] G. Mahlon, Fermilab preprint Fermilab-Pub-93/327-T (1993).
- [8] Z. Bern, L. Dixon, and D.A. Kosower, Phys. Rev. Lett. **B70** (1993), 2677; Z. Bern, L. Dixon, D. C. Dunbar, and D. A. Kosower, SLAC preprint SLAC-PUB-6415.
- [9] C.S. Lam and J.P. Lebrun, Nuovo Cimento **59A** (1969), 397.
- [10] G. Veneziano, Nuovo Cimento **57A** (1968), 190.
- [11] M. Strassler, Nucl. Phys. **B385** (1992) 145; SLAC preprint SLAC-PUB 5978 (1992).
- [12] M.G. Schmidt and C. Schubert, DESY preprint DESY-HD-THEP-94-7 (hep-ph/9403158) (1994).